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Expansive homeomorphisms in continuum theory*

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Abstract

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In this paper, we study expansive homeomorphisms from a point of view of continuum theory.

Keywords: Expansive homeomorphism, positively expansive map, shift homeomorphism of inverse limit, stable and unstable, tree-like, circle-like, crooked, indecomposable continuum.

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1. Introduction

All spaces under consideration are assumed to be metric. By a *continuum* we mean a compact connected nondegenerate space. Let X be a compact metric space with metric d . A homeomorphism $f: X \rightarrow X$ of X is called *expansive* if there is a positive number $c > 0$ (called an *expansive constant for f*) such that if x and y are different points of X , then there is an integer $n = n(x, y) \in \mathbb{Z}$ such that

$$d(f^n(x), f^n(y)) > c.$$

Expansiveness does not depend on the choice of metric d of X . In [17], Mañé proved that if $f: X \rightarrow X$ is an expansive homeomorphism of a compact metric space X , then $\dim X < \infty$ and every minimal set is 0-dimensional. This result shows that there is some restriction on which spaces admit expansive homeomorphisms. We

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are interested in the following problem [4]: What kinds of continua admit expansive homeomorphisms? In [24], Williams first showed that there is a 1-dimensional continuum admitting expansive homeomorphisms. In fact, he proved that the shift homeomorphism of the dyadic solenoid is expansive. From continuum theory in topology, we know that inverse limits spaces yield powerful techniques for constructing complicated spaces and maps from simple spaces and maps. Naturally, we are also interested in the next problem: What kinds of maps induce expansiveness of the shift homeomorphisms? It is well known that positively expansive maps induce expansiveness of the shift homeomorphisms (e.g., see [24]). In [7], Jacobson and Utz asserted that shift homeomorphisms of the inverse limit of every surjective map of an arc is not expansive (see [1, p. 648] for the complete proof). It is known that “Plykin’s attractors” are 1-dimensional continua in the plane \mathbb{R}^2 and are examples of Williams’ 1-dimensional expanding attractors, on which homeomorphisms are not only expansive homeomorphisms but even hyperbolic diffeomorphisms (see [20, 21]). Also, Plykin’s attractors can be represented as inverse limits of maps $g: K \rightarrow K$ of graphs such that the shift homeomorphisms \tilde{g} of the maps are expansive (see [20, p. 243; 21, p. 121]). In [13], we proved that if an onto map $f: G \rightarrow G$ of a graph G is null-homotopic, then the shift homeomorphism \tilde{f} of f is not expansive. In particular, shift homeomorphisms of tree-like continua are not expansive. Also, we proved that for any graph G containing a simple closed curve, there is an onto map $f: G \rightarrow G$ such that the shift homeomorphism \tilde{f} of f is expansive. Hence, there is a G -like continuum X admitting an expansive homeomorphism.

In this paper, we investigate expansive homeomorphisms from a point of view of continuum theory. In Section 2, by using the notion of positively pseudo-expansive map, we give a characterization of expansiveness of shift homeomorphisms of inverse limits of graphs. We can easily see that the characterization is not true for the case of n -dimensional polyhedra ($n \geq 2$). In Section 3, 4 and 5, we deal with more general expansive homeomorphisms which are not shift homeomorphisms. In Section 3, we prove that if $f: X \rightarrow X$ is an expansive homeomorphism of a compact metric space X with $\dim X \geq 1$, then there exists a closed subset Z of X such that each component of Z is nondegenerate, the space of components of Z is a Cantor set, the decomposition space of Z into components is continuous, and all components of Z are stable or unstable. In Section 4, we prove that there are no expansive homeomorphisms on hereditarily decomposable tree-like continua. In Section 5, we also prove that there are no expansive homeomorphisms on hereditarily decomposable circle-like continua.

2. A characterization of expansiveness of shift homeomorphisms of inverse limits of graphs

Let X be a compact metric space with metric d . By the *hyperspace* of X , we mean $C(X) = \{A \mid A \text{ is a nonempty subcontinuum of } X\}$ with the *Hausdorff metric* d_H ,

i.e., $d_H(A, B) = \inf\{\varepsilon > 0 \mid U_\varepsilon(A) \supset B \text{ and } U_\varepsilon(B) \supset A\}$, where $U_\varepsilon(A)$ denotes the ε -neighborhood of A in X . It is well known that if X is a continuum, then $C(X)$ is arcwise connected (e.g., see [18]).

Let f be an expansive homeomorphism of a compact metric space X with an expansive constant $c > 0$. If $\varepsilon > 0$, let W_ε^s and W_ε^u be the *local stable* and *unstable* families of subcontinua of X defined by

$$W_\varepsilon^s = \{A \in C(X) \mid \text{diam } f^n(A) \leq \varepsilon \text{ for any } n \geq 0\},$$

$$W_\varepsilon^u = \{A \in C(X) \mid \text{diam } f^{-n}(A) \leq \varepsilon \text{ for any } n \geq 0\}.$$

Also, define families W^s and W^u of *stable* and *unstable subcontinua* as

$$W^s = \left\{ A \in C(X) \mid \lim_{n \rightarrow \infty} \text{diam } f^n(A) = 0 \right\},$$

$$W^u = \left\{ A \in C(X) \mid \lim_{n \rightarrow \infty} \text{diam } f^{-n}(A) = 0 \right\}.$$

Then we know that if $\varepsilon \leq c$, then $W^s = \{f^{-n}(A) \mid A \in W_\varepsilon^s, n \geq 0\}$ and $W^u = \{f^n(A) \mid A \in W_\varepsilon^u, n \geq 0\}$ (see [17, p. 315]).

Let X be a compact metric space with metric d . For a map $f: X \rightarrow X$, let

$$(X, f) = \{(x_i)_{i=1}^\infty \mid x_i \in X, f(x_{i+1}) = x_i \ (i \geq 1)\}.$$

Define a metric \tilde{d} for (X, f) by

$$\tilde{d}(\tilde{x}, \tilde{y}) = \sum_{i=1}^{\infty} d(x_i, y_i)/2^i, \quad \text{where } \tilde{x} = (x_i)_{i=1}^\infty, \tilde{y} = (y_i)_{i=1}^\infty \in (X, f).$$

Then the space (X, f) is called the *inverse limit of the map $f: X \rightarrow X$* . Define a map \tilde{f} by

$$\tilde{f}((x_i)_{i=1}^\infty) = (f(x_i))_{i=1}^\infty = (x_{i-1})_{i=1}^\infty.$$

Then \tilde{f} is a homeomorphism and called the *shift homeomorphism* of f . Let $p_n: (X, f) \rightarrow X_n = X$ be the projection defined by $p_n((x_i)_{i=1}^\infty) = x_n$.

Let A be a closed subset of a compact metric space X with metric d . A map $f: X \rightarrow X$ is *positively expansive* on A if there is a positive number $c > 0$ such that if $x, y \in A$ and $x \neq y$, then there is a natural number $n \geq 0$ such that

$$d(f^n(x), f^n(y)) > c.$$

Such a positive number c is called a *positively expansive constant* for $f|A$. If $f: X \rightarrow X$ is a positively expansive map on the total space X , then f is called *positively expansive*. Clearly, if $f: X \rightarrow X$ is a positively expansive map on A , then $f|A: A \rightarrow X$ is locally injective, but we cannot conclude that $f(A) \subset A$.

Let \mathcal{A} be a finite closed covering of a compact metric space X . A map $f: X \rightarrow X$ is called a *positively pseudo-expansive map with respect to \mathcal{A}* if

- (P₁) f is positively expansive on A for each $A \in \mathcal{A}$, and
- (P₂) for the case $A, B \in \mathcal{A}$ and $A \cap B \neq \emptyset$, one of the following two conditions holds:
 - (*) f is positively expansive on $A \cup B$;
 - (**) if f is not positively expansive on $A \cup B$, then there is a natural number $k \geq 1$ such that for any $A', A'' \in \mathcal{A}$ with $A' \cap A'' \neq \emptyset$,

$$f^k(A' \cup A'') \cap (A - B) = \emptyset \quad \text{or} \quad f^k(A' \cup A'') \cap (B - A) = \emptyset.$$

A map $f: X \rightarrow X$ is called *positively pseudo-expansive* if f is positively pseudo-expansive with respect to some finite closed covering \mathcal{A} of X . This notion is important for constructing various kinds of expansive homeomorphisms (e.g., see Facts 2.1, 2.2 and 2.4). By the definitions, positively expansive maps imply positively pseudo-expansive maps, but the converse assertion is not true. Concerning positively pseudo-expansive maps of graphs (= 1-dimensional compact connected polyhedra), we know the following facts (see [13]).

Fact 2.1. *If $f: X \rightarrow X$ is a positively pseudo-expansive map of a compact metric space X , then the shift homeomorphism \tilde{f} of f is expansive.*

Fact 2.2. *Let G be a graph. Then G admits a positively pseudo-expansive map if and only if G contains a simple closed curve.*

Fact 2.3. *Let $f: G \rightarrow G$ be an onto map of a graph G . If f is null-homotopic, then the shift homeomorphism \tilde{f} is not expansive, hence f is not a positively pseudo-expansive map.*

Fact 2.4. *If $f: G \rightarrow G$ is a positively expansive map of a graph G , then the inverse limit space (G, f) of f cannot be embedded in the plane, but there are various kinds of positively pseudo-expansive maps such that the inverse limits of the maps can be embeddable in the plane.*

Now, we shall prove the following characterization of expansiveness of shift homeomorphisms of inverse limits of graphs, which is the main theorem of this section.

Theorem 2.5. *Let $f: G \rightarrow G$ be an onto map of a graph G . Then the shift homeomorphism \tilde{f} of f is expansive if and only if f is a positively pseudo-expansive map with respect to some finite closed covering \mathcal{A} of G , where $\mathcal{A} = \{e \mid e \text{ is an edge of some simplicial complex } K^* \text{ such that } |K^*| = G\}$.*

To prove Theorem 2.5, we need the following lemmas.

Lemma 2.6 [17, p. 315]. *Let $h: X \rightarrow X$ be an expansive homeomorphism of a compact metric space X and let $c > 0$ be an expansive constant for h . Then for all $\gamma > 0$ there*

is $N > 0$ such that $h^n(W_c^s(x)) \subset W_\gamma^s(h^n(x))$, $h^{-n}(W_c^u(x)) \subset W_\gamma^u(h^{-n}(x))$ for all $x \in X$ and $n \geq N$, where $W_\varepsilon^s(x) = \{y \in X \mid d(h^n(x), h^n(y)) \leq \varepsilon \text{ for all } n \geq 0\}$ and $W_\varepsilon^u(x) = \{y \in X \mid d(h^{-n}(x), h^{-n}(y)) \leq \varepsilon \text{ for all } n \geq 0\}$.

Lemma 2.7 [17, p. 318]. *Let h and c be as in Lemma 2.6. Let $0 < \varepsilon < c/2$. Then there is $\delta > 0$ such that if $x, y \in X$, $d(x, y) \leq \delta$, and for some $n > 0$, $\varepsilon \leq \sup\{d(h^j(x), h^j(y)) \mid 0 \leq j \leq n\} \leq 2\varepsilon$, then $d(h^n(x), h^n(y)) \geq \delta$.*

The following lemma is essentially the same as in [12, (2.2)], but the statement is slightly different. For completeness, we give the proof.

Lemma 2.8. *Let $h : X \rightarrow X$ be an expansive homeomorphism of a compact metric space X . Let ε and δ be as in Lemma 2.7. If A is any nondegenerate subcontinuum of X such that $\text{diam } A \leq \delta$ and $\text{diam } h^m(A) > \varepsilon$ for some $m \in \mathbb{Z}$, then one of the following two conditions holds:*

- (a) *If $m \geq 0$, then $\text{diam } h^n(A) \geq \delta$ for $n \geq m$.*
- (b) *If $m < 0$, then $\text{diam } h^{-n}(A) \geq \delta$ for $n \geq -m$.*

Proof. We shall show that δ satisfies the conditions as in Lemma 2.7. Let A be a nondegenerate subcontinuum of X with $\text{diam } A \leq \delta$. We show the case $m \geq 0$. Choose two points a and b of A such that $d(h^m(a), h^m(b)) > \varepsilon$. Let $n \geq m$. Choose $\eta > 0$ such that $d(h^j(x), h^j(y)) < \varepsilon$ for $x, y \in X$ with $d(x, y) < \eta$ and $0 \leq j \leq n$. Since A is a continuum, there is a finite sequence $a = a_0, a_1, \dots, a_p = b$ of points of A such that $d(a_i, a_{i+1}) < \eta$ for each i . For each $0 \leq r \leq p$, define $S_r = \sup\{d(h^j(a_0), h^j(a_r)) \mid 0 \leq j \leq n\}$. Then $S_0 = 0$, $S_p > \varepsilon$ and $|S_{r+1} - S_r| \leq \varepsilon$ for all r . Choose r such that $S_{r-1} \leq \varepsilon$ and $S_r > \varepsilon$. Then $S_r \leq 2\varepsilon$. By Lemma 2.7, we see that $\text{diam } h^n(A) \geq d(h^n(a_0), h^n(a_r)) \geq \delta$. The case $m < 0$ is the same as before. \square

Proof of Theorem 2.5. By Fact 2.1, it is enough to show that if \tilde{f} is expansive, then f is a positively pseudo-expansive map. Suppose that \tilde{f} is expansive and $c > 0$ be an expansive constant for \tilde{f} . Let $X = (G, f)$.

First, we shall show that there is a positive number $\alpha > 0$ such that if A is a subcontinuum of X with $\text{diam } A \leq \alpha$, then $A \in W_c^u$. Suppose, on the contrary, that for each $n = 1, 2, \dots$, there are subcontinua A_n of X such that

$$\text{diam } A_n \leq 1/n, \tag{1}$$

and

$$A_n \text{ is not contained in } W_c^u. \tag{2}$$

Since $C(X)$ is a compact metric space, we may assume that

$$\lim_{n \rightarrow \infty} A_n = \{x\} \quad \text{for some } x \in X. \tag{3}$$

Let $\delta > 0$ be as in Lemma 2.8. We may assume that $\text{diam } A_n \leq \delta$ for all n . Since A_1 is not contained in W_c^u , there is a natural number $m(1) > 0$ such that $\text{diam } \tilde{f}^{-m(1)}(A_1) > c > 2\varepsilon$. Choose a sufficiently small neighborhood U_1 of x in X such that there is a subcontinuum B_1 of A_1 such that $\text{diam } \tilde{f}^{-m(1)}(B_1) \geq \varepsilon$ and $B_1 \cap U_1 = \emptyset$. By (3), we can choose $n(2)$ such that $A_{n(2)} \subset U_1$. Since $A_{n(2)}$ is not contained in W_c^u , there is a natural number $m(2) > 0$ such that $\text{diam } \tilde{f}^{-m(2)}(A_{n(2)}) > c > 2\varepsilon$. Choose a sufficiently small neighborhood U_2 of x in X such that $U_2 \cap B_1 = \emptyset$ and for some subcontinuum $B_{n(2)}$ of $A_{n(2)}$, $\text{diam } \tilde{f}^{-m(2)}(B_{n(2)}) \geq \varepsilon$ and $B_{n(2)} \cap U_2 = \emptyset$. If we continue this procedure, we obtain two sequences $\{m(i)\}_{i=1}^\infty$ and $1 = n(1) < n(2) < \dots$, of natural numbers, and a sequence $\{B_{n(i)}\}$ of subcontinua of X such that

$$B_{n(i)} \cap B_{n(j)} = \emptyset \quad (i \neq j), \quad (4)$$

and

$$\text{diam } \tilde{f}^{-m(i)}(B_{n(i)}) \geq \varepsilon \quad \text{for each } i = 1, 2, \dots \quad (5)$$

By Lemma 2.8 and (5), we see that

$$\text{diam } \tilde{f}^{-n}(B_{n(i)}) \geq \delta \quad \text{for all } n \geq m(i). \quad (6)$$

We can choose a sufficiently large natural number n_0 and a sufficiently small positive number $\eta = \eta(n_0, \delta) > 0$ such that if E is a subset of $X = (G, f)$ and $\text{diam } E \geq \delta$, then $\text{diam } p_{n_0}(E) \geq \eta$. Also, choose a natural number m_0 such that if E_1, E_2, \dots, E_{m_0} are subcontinua of G and $\text{diam } E_i \geq \eta$ ($i = 1, 2, \dots, m_0$), then for some i and j ($i \neq j$) $E_i \cap E_j \neq \emptyset$. Consider the family $\mathcal{B} = \{B_{n(i)} \mid i = 1, 2, \dots, m_0\}$. By (4), we can choose a natural number $n_1 > n_0$ such that $p_n(B_{n(i)}) \cap p_n(B_{n(j)}) = \emptyset$ for $1 \leq i < j \leq m_0$ and $n \geq n_1$. Put $N = \max\{n_1, m_0, m(i) \mid i = 1, 2, \dots, m_0\}$. By (6), we see that

$$\text{diam } \tilde{f}^{-N}(B_{n(i)}) \geq \delta \quad \text{for } i = 1, 2, \dots, m_0. \quad (7)$$

Hence, for $i = 1, 2, \dots, m_0$,

$$\text{diam } p_{n_0+N}(B_{n(i)}) = \text{diam } p_{n_0}(\tilde{f}^{-N}(B_{n(i)})) \geq \eta, \quad (8)$$

and

$$p_{n_0+N}(B_{n(i)}) \cap p_{n_0+N}(B_{n(j)}) = \emptyset \quad (i \neq j). \quad (9)$$

This is a contradiction. Therefore there exists $\alpha > 0$ such that if $A \in C(X)$ and $\text{diam } A \leq \alpha$, then $A \in W_c^u$.

Consider the following set $C(X, \alpha) = \{A \in C(X) \mid \text{diam } A = \alpha\}$. Then $C(X, \alpha)$ is closed in $C(X)$ and if $x \in X$, there is some $A \in C(X, \alpha)$ such that $x \in A$, because the segment from $\{x\}$ to X in $C(X)$ is an arc (see [18, (1.18)]). By Lemma 2.6, for any $\gamma > 0$, there is N such that $\text{diam } \tilde{f}^{-n}(A) < \gamma$ for all $A \in C(X, \alpha)$ and $n \geq N$. Hence we can choose a natural number N such that $p_n(A)$ is a tree (= a graph containing no simple closed curves) for all $A \in C(X, \alpha)$ and $n \geq N - 1$. Since \tilde{f} is expansive and $p_n(A)$ is a tree ($n \geq N - 1$), we see that $f|_{p_{n+1}(A)}: p_{n+1}(A) \rightarrow p_n(A)$ is a homeomorphism for all $A \in C(X, \alpha)$ and $n \geq N - 1$ (see the proof of [13, (3.5)]). We may assume that $\text{diam } p_N(A) \geq \beta > 0$ for some $\beta > 0$ and all $A \in C(X, \alpha)$. Since p_N is an onto map, we can easily see that for any $x \in G$, there is an arc A_x in G

such that A_x contains x , $\text{diam } A_x \geq \beta$, $f|_{A_x}: A_x \rightarrow G$ is a positively expansive map and $A_x \subset p_N(A)$ for some $A \in C(X, \alpha)$. Note that if $x_p \in G$, $\lim_{p \rightarrow \infty} x_p = x \in G$ and $\lim_{p \rightarrow \infty} A_{x_p} = B$, then $x \in B$ and $B \subset p_N(A)$ for some $A \in C(X, \alpha)$. By using this fact, we can easily see that there is a simplicial complex K such that $|K| = G$ and if E is any edge of K , then for some $A \in C(X, \alpha)$, $p_N(A) \supset E$. Let $k = \#\{(E, E') \mid E, E' \in K^1, E \cap E' \neq \emptyset \text{ and } E \neq E'\}$, where $\#Y$ denotes the cardinal number of a set Y and K^1 denotes the 1-skeleton of K . Choose a subdivision K^* of K such that $f^j(e \cup e')$ is a tree for $e, e' \in K^{*1}$ with $e \cap e' \neq \emptyset$ and $0 \leq j \leq k$ and if $f^j|_{e \cup e'}$ is not locally injective at the point p ($e \cap e' = \{p\}$), $f^j(e \cup e') \subset E$ for some $E \in K^1$ and $0 \leq j \leq k$.

Finally, we shall prove that f is a positively pseudo-expansive map with respect to $\mathcal{A} = \{e \mid e \text{ is an edge of } K^*\}$. Note that $f^j|_e: e \rightarrow G$ is injective for each $e \in \mathcal{A}$ and $0 \leq j \leq k$, because $f^j(e)$ is a tree (more precisely an arc). Clearly, \mathcal{A} satisfies the condition (P_1) . We must show that \mathcal{A} satisfies (P_2) . Suppose that for some $e_1, e_2 \in K^{*1}$ with $e_1 \neq e_2$ and $e_1 \cap e_2 \neq \emptyset$, f is not positively expansive on $e_1 \cup e_2$. Let $e_1 \cap e_2 = \{p\}$. Clearly, $f^m|_{e_1 \cup e_2}$ is not locally injective at the point p for some $m > 0$. Suppose, on the contrary, that there are e'_1 and e'_2 of \mathcal{A} ($e'_1 \neq e'_2$) such that $e'_1 \cap e'_2 = \{p'\} \neq \emptyset$ and

$$f^k(e'_1 \cup e'_2) \cap (e_1 - e_2) \neq \emptyset \neq f^k(e'_1 \cup e'_2) \cap (e_2 - e_1). \quad (10)$$

Note that $f^k(p') = p$, because \tilde{f} is expansive (see the proof of [13, (3.5)]). Also, we can see that $f^j(p') \in K^0$ and $f^j|_{e'_1 \cup e'_2}$ is locally injective at p' for $0 \leq j \leq k$. By the definition of k , we see that for some $j(1)$ and $j(2)$ with $0 \leq j(1) < j(2) \leq k$, $f^{j(2)}(p') = f^{j(1)}(p')$ and $(f^{j(2)}(e'_1) \cap f^{j(1)}(e'_1)) - \{f^{j(1)}(p')\} \neq \emptyset$, $(f^{j(2)}(e'_2) \cap f^{j(1)}(e'_2)) - \{f^{j(1)}(p')\} \neq \emptyset$. Hence we can conclude that $f^n|_{e'_1 \cup e'_2}$ is locally injective at p' for all $n \geq 0$. This is a contradiction. Hence \mathcal{A} satisfies (P_2) . This completes the proof. \square

Corollary 2.9. *Let $f: G \rightarrow G$ be an onto map of a graph G . If the shift homeomorphism \tilde{f} of f is expansive, then there is a positive number $\alpha > 0$ such that if $A \in C((G, f))$ and $\text{diam } A \leq \alpha$, $A \in W^u = \{D \in C((G, f)) \mid \lim_{n \rightarrow \infty} \text{diam } \tilde{f}^{-n}(D) = 0\}$. Also, for any $x \in (G, f)$, there is an arc A in (G, f) containing x such that $A \in W^u$.*

Corollary 2.10. *Let $f: G \rightarrow G$ be an onto map of a graph G . If \tilde{f} is an expansive homeomorphism, then $W^s = \{\{x\} \mid x \in (G, f)\}$, where*

$$W^s = \{A \in C((G, f)) \mid \lim_{n \rightarrow \infty} \text{diam } \tilde{f}^n(A) = 0\}.$$

Remark 2.11. In the statement of Theorem 2.5, the cases of n -dimensional polyhedra ($n \geq 2$) are not true. It is well known that there is an expansive homeomorphism f of the 2-torus $T = S^1 \times S^1$ [22]. The shift homeomorphism \tilde{f} of f is topologically conjugate to f . Note that if $f: X \rightarrow X$ is an expansive homeomorphism of a Peano continuum X , then for any open set U , $f|_U$ is not positively expansive. Then \tilde{f} is expansive, but f is not positively pseudo-expansive.

Remark 2.12. There is an expansive homeomorphism F of a 1-dimensional continuum X such that F and F^{-1} cannot be represented by shift homeomorphisms of maps of graphs. Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, where \mathbb{C} is the set of complex numbers and let $f: S^1 \rightarrow S^1$ be a map defined by $f(e^{i\theta}) = e^{i2\theta}$. Then \tilde{f} is an expansive homeomorphism of the dyadic solenoid (S^1, f) [24]. Let p be the fixed point of \tilde{f} . Let $D = (S^1, f)$ and let $X = (D_1, p_1) \vee (D_2, p_2)$ be the one-point union of two copies of (D, p) . Define a map $F: X \rightarrow X$ by $F(x) = \tilde{f}(x)$ for $x \in D_1$ and $F(x) = \tilde{f}^{-1}(x)$ for $x \in D_2$. By Corollary 2.9, F is a desired expansive homeomorphism.

Example 2.13. In [20, 21], it was shown that for each $n = 3, 4, 5, \dots$, there exist a graph G_n and an onto map $g_n: G_n \rightarrow G_n$ such that

$$\pi_1(G_n) = \underbrace{\mathbb{Z} * \mathbb{Z} * \dots * \mathbb{Z}}_n,$$

g_n is a homotopy equivalence and the shift homeomorphism \tilde{g}_n of g_n is expansive (hence, g_n is a positively pseudo-expansive map). Here, we give an example which implies that the case $n = 2$ is also true. Let G_2 be the one-point union of two oriented circles \vec{a} and \vec{b} . Note that $\pi_1(G_2) = \mathbb{Z} * \mathbb{Z}$. Define a map $g_2: G_2 \rightarrow G_2$ by

$$\begin{aligned} \vec{a} &\longrightarrow \vec{a} * \vec{b} * \vec{a}, \\ \vec{b} &\longrightarrow \vec{b} * \vec{a}. \end{aligned}$$

Then we can easily see that g_2 is a positively pseudo-expansive map and g_2 is a homotopy equivalence. In fact, the homotopy inverse $h: G_2 \rightarrow G_2$ is defined by

$$\begin{aligned} \vec{a} &\longrightarrow \vec{a} * (\vec{b})^{-1}, \\ \vec{b} &\longrightarrow \vec{b} * \vec{b} * (\vec{a})^{-1}. \end{aligned}$$

Also, we can easily see that the case $n = 1$ is not true, i.e., for any graph G_1 with $\pi_1(G_1) = \mathbb{Z}$, there are no positively pseudo-expansive maps which are homotopy equivalences. The case $n = 0$ is not true. In fact, there are no positively pseudo-expansive maps on trees [13].

3. Stable and unstable subcontinua of expansive homeomorphisms

In [11], we proved that there are no expansive homeomorphisms on Suslinian continua. In this section, we give a more precise result which is related to the stable and unstable properties. The next result is the main theorem of this section.

Theorem 3.1. *Let X be a compact metric space with $\dim X \geq 1$. If $f: X \rightarrow X$ is an expansive homeomorphism, then there is a closed subset Z of X such that each component of Z is nondegenerate, the space of components of Z is a Cantor set, the decomposition space of Z into components is continuous (i.e., upper semi- and lower semicontinuous), and all components of Z are contained in W^s or W^u .*

The proof is similar to [11, Theorem (3.1)], but we need more precise information. Let c be an expansive constant for f . Fix $0 < \varepsilon < c/2$. By [17, p. 315], one of $W_{2\varepsilon}^s$ and $W_{2\varepsilon}^u$ has a nondegenerate element. From now on, we assume that $W_{2\varepsilon}^u$ has a nondegenerate element. Let M be a nonempty subset of $W_{2\varepsilon}^u$. Define M^f by $M^f = \{A \in C(X) \mid \text{for any } \eta > 0 \text{ and any natural number } k, \text{ there are points } A_1, A_2, \dots, A_k \text{ of } M \text{ such that each } A_i \text{ is nondegenerate, } A_i \cap A_j = \emptyset \ (i \neq j) \text{ and } d_H(A, A_i) < \eta \text{ for } i = 1, 2, \dots, k\}$.

Then we have

Proposition 3.2. (1) *The set $W_{2\varepsilon}^u$ is closed in $C(X)$.*

(2) *M^f is a closed subset of $W_{2\varepsilon}^u$.*

(3) *$M^f \supset (M^f)^f$.*

For ordinal numbers, define $M_0 = M$, $M_1 = M^f$, $M_{\alpha+1} = (M_\alpha)^f$ and $M_\lambda = \bigcap_{\alpha < \lambda} M_\alpha$, where λ is a limit ordinal number.

The proof of the following proposition is similar to [11, (3.4)]. For completeness we give the proof.

Proposition 3.3. *Let $M = W_{2\varepsilon}^u$. Then $M_\lambda \neq \emptyset$ for any countable ordinal number λ if and only if there is a closed subset Z of X such that each component of Z is nondegenerate, the space of components of Z is a Cantor set, the decomposition space of Z into components is continuous, and all components of Z are elements of $W_{2\varepsilon}^u$.*

Proof. Suppose that $M_\lambda \neq \emptyset$ for any countable ordinal number λ . By Proposition 3.2, M_λ is closed in $C(X)$ and $M_\alpha \supset M_\beta$ if $\alpha < \beta$. Since $C(X)$ is separable, there is a countable ordinal number α such that $M_\alpha = M_\beta$ for any $\beta \geq \alpha$. In particular, $M_\alpha = M_{\alpha+1} \neq \emptyset$. Choose $A \in M_\alpha$. Since $A \in (M_\alpha)^f$, there are two points A_0 and A_1 of M_α such that each A_i is nondegenerate, $A_0 \cap A_1 = \emptyset$. Choose $\gamma > 0$ such that $\text{diam } A_i > \gamma$ ($i = 0, 1$), and choose neighborhoods U_i ($i = 0, 1$) of A_i in X such that $\text{Cl } U_0 \cap \text{Cl } U_1 = \emptyset$ and $\text{Cl } U_i \subset U_{1/2}(A_i)$. Since A_i is contained in $M_\alpha = (M_\alpha)^f$, for each i we can choose two points A_{ij} ($j = 0, 1$) of M_α such that $d_H(A_i, A_{ij}) < 1/2^2$ ($j = 0, 1$), $\text{diam } A_{ij} > \gamma$, $A_{i0} \cap A_{i1} = \emptyset$ and $A_{ij} \subset U_i$. Choose neighborhoods U_{ij} of A_{ij} in U_i such that $\text{Cl } U_{i0} \cap \text{Cl } U_{i1} = \emptyset$ and $\text{Cl } U_{ij} \subset U_{(1/2^2)}(A_{ij})$. Note that $A_{ij} \in M_\alpha = (M_\alpha)^f$. By induction on n , we can choose subcontinua $A_{i_1 i_2 \dots i_n}$ ($i_k = 0$ or 1) of X and neighborhoods $U_{i_1 i_2 \dots i_n}$ of $A_{i_1 i_2 \dots i_n}$ in $U_{i_1 i_2 \dots i_{n-1}}$ such that

(1) $d_H(A_{i_1 i_2 \dots i_{n-1}}, A_{i_1 i_2 \dots i_{n-1} j}) < 1/2^n$ ($j = 0, 1$),

(2) $\text{Cl } U_{i_1 i_2 \dots i_{n-1} 0} \cap \text{Cl } U_{i_1 i_2 \dots i_{n-1} 1} = \emptyset$,

(3) $\text{diam } A_{i_1 i_2 \dots i_n} > \gamma$, and

(4) $\text{Cl } U_{i_1 i_2 \dots i_n} \subset U_{(1/2^n)}(A_{i_1 i_2 \dots i_n})$.

For any sequence $\{(i_k)_n\}$ ($i_k = 0$ or 1), consider the following set

$$A_{i_1 i_2 \dots} = \bigcap_{n=1}^{\infty} \text{Cl } U_{i_1 i_2 \dots i_n}.$$

Set $Z = \bigcup \{A_{i_1 i_2 \dots} \mid i_j = 0 \text{ or } 1\}$. Then the set of components of Z is $\{A_{i_1 i_2 i_3 \dots} \mid i_k = 0 \text{ or } 1\}$. Since $M = W_{2^e}^u$ is closed in $C(X)$, $A_{i_1 i_2 \dots} \in W_{2^e}^u$. Clearly, Z satisfies the desired conditions.

Next, suppose that there is a closed set Z as in Proposition 3.3. Clearly, each component C of Z is an element of M_λ for any (countable) ordinal number λ . Hence $M_\lambda \neq \emptyset$ for any countable ordinal number λ . This completes the proof. \square

The following lemma is trivial.

Lemma 3.4. *Let Y be a compact metric space. Let $\eta > 0$ and k be any natural number. Then there is a natural number $n = n(\eta, k) \geq k$ such that if a_1, a_2, \dots, a_n are points of Y , then there is a point a of Y such that $d(a, a_{i_j}) < \eta$ for $j = 1, 2, \dots, k$, where $1 \leq i_1 < i_2 < \dots < i_k \leq n$.*

From now on, suppose that $M = W_{2^e}^u$.

By Lemma 3.4 and the transfinite induction, we can see the following

Lemma 3.5. *Let λ be a countable ordinal number. If $A \in M_\lambda$ and A is nondegenerate, for any open sets U, V of X such that $\text{Cl } V \subset U$, $A \cap V \neq \emptyset$ and $A - \text{Cl } U \neq \emptyset$, there exist $B \in M_\lambda$ such that $B \cap \text{Cl } V \neq \emptyset$, $B \subset A \cap \text{Cl } U$, and $B \cap \text{Bd } U \neq \emptyset$.*

Suppose that Y is a continuum with $\text{diam } Y \geq t \geq 0$. Then there is a subcontinuum A of Y such that $\text{diam } A = t$. By this fact, Lemma 3.4 and the transfinite induction, we can see the following

Lemma 3.6. *Suppose that λ is a countable ordinal number, $0 < t \leq 2\varepsilon$, $A \in M_\lambda$ and $\text{diam } f(A) \geq t$. Then there is a subcontinuum B of A such that $f(B) \in M_\lambda$ and $\text{diam } f(B) = t$.*

Lemma 3.7. *Let $A \in M_\lambda$, where λ is a countable ordinal number. If $\text{diam } f^j(A) < 2\varepsilon$ for $0 < j \leq m$, then $f^m(A) \in M_\lambda$.*

By Lemma 2.8, we can easily see the following

Lemma 3.8 [12, (2.2)]. *Let $\delta > 0$ be as in Lemma 2.8. Then for each nondegenerate subcontinuum A of X , there is a natural number n_0 satisfying one of the following two conditions:*

- (*) $\text{diam } f^n(A) \geq \delta$ for $n \geq n_0$;
- (**) $\text{diam } f^{-n}(A) \geq \delta$ for $n \geq n_0$.

From now, we shall give the proof of Theorem 3.1. Set $\delta_1 = \min\{\sigma, \varepsilon\}$. Choose a sequence $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \dots$ of positive numbers such that $\lim_{i \rightarrow \infty} \varepsilon_i = 0$. For each ε_k and k , choose a natural number $n_k = n(\varepsilon_k, k)$ as in Lemma 3.4. Let $A \in M = W_{2\varepsilon}^u$ and A is nondegenerate. By Lemma 3.5, we can choose nondegenerate subcontinua B_1, B_2, \dots, B_{n_k} of A such that $B_i \cap B_j = \emptyset$ ($i \neq j$) and $\text{diam } B_i \leq \delta_1$. Note that $B_i \in M = W_{2\varepsilon}^u$. By Corollary 2.9, we choose a natural number $m(k)$ such that

$$\text{diam } f^{m(k)}(B_i) \geq \delta_1 \quad \text{for each } i = 1, 2, \dots, n_k. \quad (11)$$

For each $i = 1, 2, \dots, n_k$, we define a subcontinuum C_i of B_i as follows: If $\text{diam } f^j(B_i) < 2\varepsilon$ for $0 \leq j \leq m(k)$, set $C_i = B_i$. Then by Lemma 3.7, $f^{m(k)}(C_i) \in M$. Suppose that for some $0 < m \leq m(k)$, $\text{diam } f^m(B_i) \geq 2\varepsilon$ and $\text{diam } f^j(B_i) < 2\varepsilon$ for $0 \leq j < m$. By Lemma 3.6, we choose a subcontinuum B_i^1 of B_i such that $\text{diam } f^m(B_i^1) = 2\varepsilon$ and $f^m(B_i^1) \in M$. If $\text{diam } f^j(B_i^1) < 2\varepsilon$ for $m < j \leq m(k)$, set $C_i = B_i^1$. Then $f^{m(k)}(C_i) \in M$. If for some m' ($m < m' \leq m(k)$), $\text{diam } f^{m'}(B_i^1) \geq 2\varepsilon$ and $\text{diam } f^j(B_i^1) < 2\varepsilon$ for $m < j < m'$, we choose a subcontinuum B_i^2 of B_i^1 such that $\text{diam } f^{m'}(B_i^2) = 2\varepsilon$ and $f^{m'}(B_i^2) \in M$. Moreover, if $\text{diam } f^j(B_i^2) < 2\varepsilon$ for $m' < j \leq m(k)$, set $C_i = B_i^2$. If we continue this procedure, we obtain a subcontinuum C_i of B_i . Then we can see that $\text{diam } f^j(C_i) \leq 2\varepsilon$ for $0 \leq j \leq m(k)$, $\text{diam } f^{m(k)}(C_i) \geq \delta_1$ and $f^{m(k)}(C_i) \in M$ for $i = 1, 2, \dots, n_k$ (see Lemmas 2.7 and 3.8). By the choice of n_k , there is a point $D^k \in C(X)$ such that

$$d_H(D^k, f^{m(k)}(C_{i_j})) < \varepsilon_k \quad \text{for } j = 1, 2, \dots, k, \\ \text{where } 1 \leq i_1 < i_2 < \dots < i_k \leq n_k. \quad (12)$$

Since $C(X)$ is compact, we may assume that the sequence $\{D^k\}$ is convergent to a point A_1 of $C(X)$. Then we see that $A_1 \in M_1$ and $\text{diam } A_1 \geq \delta_1$.

For a countable ordinal number λ , we may assume that for $\alpha < \lambda$, M_α contains a nondegenerate element A_α with $\text{diam } A_\alpha \geq \delta_1$. We shall prove that M_λ has the same property as M_α . Consider the two cases.

Case 1: $\lambda = \alpha + 1$. By Lemma 3.5, we can choose nondegenerate subcontinua B_1, B_2, \dots, B_{n_k} of A_α such that $\text{diam } B_i \leq \delta_1$, $B_i \cap B_j = \emptyset$ ($i \neq j$) and $B_i \in M_\alpha$. Since $B_i \in W_{2\varepsilon}^u$, there is a natural number $m(k)$ such that $\text{diam } f^{m(k)}(B_i) \geq \delta_1$ for each $i = 1, 2, \dots, n_k$. For each i , choose a subcontinuum C_i of B_i as follows. First, if $\text{diam } f^j(B_i) < 2\varepsilon$ for $0 \leq j \leq m(k)$, set $C_i = B_i$. If for some m such that $0 < m \leq m(k)$, $\text{diam } f^m(B_i) \geq 2\varepsilon$ and $\text{diam } f^j(B_i) < 2\varepsilon$ for $0 \leq j < m$, by Lemma 3.6 we can choose a subcontinuum B_i^1 of B_i such that $B_i^1 \in M_\alpha$ and $\text{diam } f^m(B_i^1) = 2\varepsilon$ and $f^m(B_i^1) \in M_\alpha$. Moreover, if $\text{diam } f^j(B_i^1) < 2\varepsilon$ for $m < j \leq m(k)$, set $C_i = B_i^1$. By Lemma 3.7, we can see that $f^{m(k)}(C_i) \in M_\alpha$. If for some m' ($m < m' < m(k)$), $\text{diam } f^{m'}(B_i^1) \geq 2\varepsilon$, we continue this procedure. Consequently, we obtain $f^{m(k)}(C_1), f^{m(k)}(C_2), \dots, f^{m(k)}(C_{n_k}) \in M_\alpha$ such that each C_i is nondegenerate, $C_i \cap C_j = \emptyset$ ($i \neq j$). Note that $\text{diam } f^{m(k)}(C_i) \geq \delta_1$ for $i = 1, 2, \dots, n_k$ (see Lemmas 2.7 and 3.8). As before, we can see that there exists $A_\lambda \in M_\lambda$ such that $\text{diam } A_\lambda \geq \delta_1$.

Case 2: λ is a limit ordinal number. In this case, choose a sequence $\alpha_1 < \alpha_2 < \alpha_3 < \dots$, of countable ordinal numbers such that $\lambda = \lim_{i \rightarrow \infty} \alpha_i$. By the inductive

hypothesis, there exist $A_i \in M_{\alpha_i}$ such that $\text{diam } A_i \geq \delta_1$. We may assume that $\{A_i\}$ is convergent to a point A_λ of $C(X)$. Then we can see that $A_\lambda \in M_\lambda$ and $\text{diam } A_\lambda \geq \delta_1$.

Consequently, we see that $M_\lambda \neq \emptyset$ for any countable ordinal number λ . By Proposition 3.3, X has a closed subset Z satisfying the desired conditions. This completes the proof. \square

Corollary 3.9. *Suppose that $f: X \rightarrow X$ is an expansive homeomorphism of a compact metric space X . If W^s (respectively W^u) contains a nondegenerate element then W^s (respectively W^u) has an uncountable family of mutually disjoint, nondegenerate subcontinua of X .*

4. The nonexistence of expansive homeomorphisms of hereditarily decomposable tree-like continua

In [13], we showed that there are various kinds of shift homeomorphisms of inverse limits of graphs which are expansive homeomorphisms. If a graph G contains a simple closed curve, then there is a positively pseudo-expansive map $f: G \rightarrow G$, in particular, the shift homeomorphism \tilde{f} of f is expansive. But, all shift homeomorphisms of tree-like continua are not expansive. Naturally, we are interested in the following problem: Is there a tree-like continuum which admits an expansive homeomorphism? In [12], we showed that there are no expansive homeomorphisms on dendroids (= arcwise connected tree-like continua). In this section, we give a more general answer. The following theorem is the main result of this section.

Theorem 4.1. *There are no expansive homeomorphisms on hereditarily decomposable tree-like continua.*

Before giving the proof of Theorem 4.1, we need some notations as follows. A continuum X is *decomposable* if X is the union of two subcontinua different from X . A continuum X is *indecomposable* if X is not decomposable. A continuum X is *hereditarily decomposable* (respectively *hereditarily indecomposable*) if each nondegenerate subcontinuum of X is decomposable (respectively indecomposable). We say X is a *tree* if X is homeomorphic to a graph which contains no simple closed curve. A continuum X is *tree-like* (respectively *arc-like*) if for each $\varepsilon > 0$, there is a finite open cover \mathcal{U} of X such that $\text{mesh } \mathcal{U} < \varepsilon$ and the nerve $N(\mathcal{U})$, if nondegenerate, is a tree (respectively an arc), where $\text{mesh } \mathcal{U} = \sup\{\delta(U) \mid U \in \mathcal{U}\}$ and $\delta(U) = \sup\{d(x, y) \mid x, y \in U\}$. A continuum X is a *dendroid* if X is an arcwise connected tree-like continuum. It is well known that every dendroid is hereditarily decomposable. By a *refinement* of a finite collection \mathcal{U} of subsets of a space X we mean, as usual, any finite collection of subsets of X whose elements are contained in elements of \mathcal{U} . Let C_1, C_2, \dots, C_m be subsets of a space X . Then the sequence is said to be a *chain*, and is denoted by $[C_1, C_2, \dots, C_m]$, provided that $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leq 1$ for each $1 \leq i, j \leq m$. A chain $[C_1, C_2, \dots, C_m]$ is said to be an η -chain

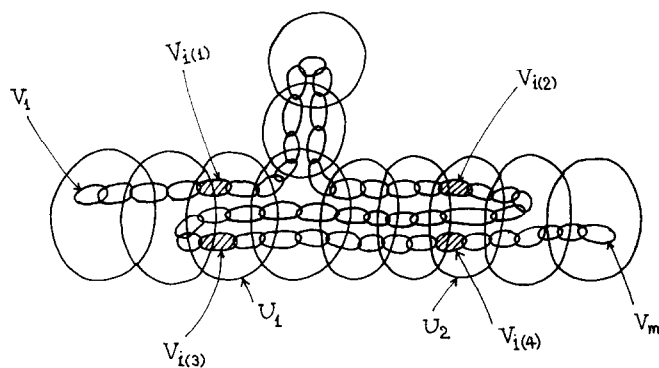


Fig. 1.

($\eta > 0$) if $\delta(C_i) < \eta$ for each $i = 1, 2, \dots, m$. Let $\delta > 0$ and n be a natural number. A chain $[C_1, C_2, \dots, C_m]$ is said to be (δ, n) -folding if there are elements $C_{i(1)}, C_{j(1)}, C_{i(2)}, C_{j(2)}, \dots, C_{i(n)}, C_{j(n)}$ of $\{C_1, C_2, \dots, C_m\}$ such that $1 \leq i(1) < j(1) \leq i(2) < j(2) \leq \dots \leq i(n) < j(n) \leq m$ and $d(C_{i(k)}, C_{j(k)}) \geq \delta$ for each $k = 1, 2, \dots, n$, where $d(A, B) = \inf\{d(a, b) \mid a \in A \text{ and } b \in B\}$. Let $[V_1, V_2, \dots, V_m]$ be a chain such that $\mathcal{V} = \{V_i \mid i = 1, 2, \dots, m\}$ is a refinement of a finite open cover \mathcal{U} of a space X . Let $U_1, U_2 \in \mathcal{U}$. Then the chain $[V_1, V_2, \dots, V_m]$ is said to be *crooked between U_1 and U_2* if there are $1 \leq i(1) < i(2) < i(3) < i(4) \leq m$ such that $V_{i(1)} \subset U_1$, $V_{i(2)} \subset U_2$, $V_{i(3)} \subset U_1$ and $V_{i(4)} \subset U_2$ (see Fig. 1). A chain $[V_1, V_2, \dots, V_m]$ is said to be a *chain from x to y* if $x \in V_1$ and $y \in V_m$.

Next, we list some facts which will be needed in the sequel. A homeomorphism f considered in this section is an expansive homeomorphism of a compact metric space X and $c > 0$ is an expansive constant for f . Fix $0 < \varepsilon < c/2$.

Lemma 4.2 [17, p. 318]. *For all $\rho > 0$ there exists a natural number $N = N(\rho)$ such that $d(x, y) \geq \rho$ implies that $\sup\{d(f^n(x), f^n(y)) \mid |n| \leq N\} > \varepsilon$.*

Lemma 4.3 (cf. Lemma 2.8). *Suppose that X is a continuum. Then there exists $\delta_1 > 0$ such that for any $\rho > 0$ there exist a natural number N and $\eta > 0$ such that if \mathcal{U} is any finite open cover of X with $\text{mesh } \mathcal{U} < \eta$ and $[U_1, U_2, \dots, U_m]$ is a chain of \mathcal{U} with $d(U_1, U_m) \geq \rho$, then one of the following conditions holds:*

- (1) $d(f^N(U_1), f^N(U_r)) \geq \delta_1$ for some $1 < r \leq m$;
- (2) $d(f^{-N}(U_1), f^{-N}(U_r)) \geq \delta_1$ for some $1 < r \leq m$.

Proof. By Lemma 2.7, there exist $\delta > 0$ satisfying the condition as in Lemma 2.7. Put $\delta_1 = \delta/3$ and $\rho' = \min\{\rho, \delta_1\}$. By Lemma 4.2, there exists $N = N(\rho')$ such that $d(x, y) \geq \rho'$ implies that $\sup\{d(f^j(x), f^j(y)) \mid |j| \leq N\} > \varepsilon$. Choose a sufficiently small positive number $\eta > 0$ such that $d(x, y) < \eta$ implies that $d(f^j(x), f^j(y)) < \min\{\varepsilon, \rho'\}$ for each $|j| \leq N$. Suppose that \mathcal{U} is any finite open cover of X with $\text{mesh } \mathcal{U} < \eta$ and $[U_1, U_2, \dots, U_m]$ is a chain of \mathcal{U} with $d(U_1, U_m) \geq \rho$. Then we

can choose a sequence p_1, p_2, \dots, p_m of points of X such that $p_i \in U_i$ for each i and $d(p_i, p_{i+1}) < \eta$ for $i = 1, 2, \dots, m-1$. Since $d(p_1, p_m) \geq \rho$, we can choose $1 < m' \leq m$ such that $d(p_1, p_k) < \rho'$ for each $1 \leq k < m'$ and $d(p_1, p_{m'}) \geq \rho'$. Then $d(p_1, p_k) < \rho' \leq \delta_1 < \delta$ for each $1 \leq k < m'$ and $d(p_1, p_{m'}) \leq d(p_1, p_{m'-1}) + d(p_{m'-1}, p_{m'}) < \rho' + \rho' \leq 2\delta_1 < \delta$. Since $d(p_1, p_{m'}) \geq \rho'$,

$$\sup\{d(f^j(p_1), f^j(p_{m'})) \mid |j| \leq N\} > \varepsilon.$$

Suppose that $\sup\{d(f^j(p_1), f^j(p_{m'})) \mid 0 \leq j \leq N\} > \varepsilon$. For each $1 \leq r \leq m'$, define

$$S_r = \sup\{d(f^j(p_1), f^j(p_r)) \mid 0 \leq j \leq N\}.$$

Then $S_1 = 0$ and $S_{m'} > \varepsilon$. Also, note that $|S_{r+1} - S_r| \leq \varepsilon$ for each $r = 1, 2, \dots, m'-1$. Hence we can choose r such that $S_{r-1} \leq \varepsilon$ and $S_r > \varepsilon$. Then $\varepsilon < S_r < 2\varepsilon$. By Lemma 2.7, $d(f^N(p_1), f^N(p_r)) \geq \delta$. Then we see

$$\begin{aligned} d(f^N(U_1), f^N(U_r)) &\geq d(f^N(p_1), f^N(p_r)) - 2 \cdot \text{mesh } f^N(\mathcal{U}) \\ &\geq \delta - 2\rho' \geq \delta - 2\delta_1 = \delta_1. \end{aligned}$$

The case of $\sup\{d(f^j(p_1), f^j(p_{m'})) \mid -N \leq j < 0\} > \varepsilon$ will be similarly proved. This completes the proof. \square

Lemma 4.4. *Suppose that X is a continuum. Let $x, y \in X$ and $x \neq y$. Then there exists $\delta_1 > 0$ such that for any $\gamma > 0$ and any natural number n , there exist a natural number N and $\eta > 0$ satisfying that if $[U_1, U_2, \dots, U_m]$ is an η -chain from x to y , then one of the following conditions holds:*

- (1) $[f^N(U_1), f^N(U_2), \dots, f^N(U_m)]$ is a γ -chain and is (δ_1, n) -folding;
- (2) $[f^{-N}(U_1), f^{-N}(U_2), \dots, f^{-N}(U_m)]$ is a γ -chain and is (δ_1, n) -folding.

Proof. Let δ_1 be a positive number as in Lemma 4.3. Put $\rho = d(x, y)/(2n+1)$. By Lemma 4.3, there is a natural number N and $n > 0$ satisfying the conditions as in Lemma 4.3. We may assume that $d(x', y') \leq \eta$ implies that $d(f^N(x'), f^N(y')) < \gamma$ and $d(f^{-N}(x'), f^{-N}(y')) < \gamma$. Suppose that $[U_1, U_2, \dots, U_m]$ is any η -chain from x to y . Since we can choose $\eta > 0$ sufficiently small, we may assume that there exists a sequence $1 \leq i(1) < j(1) \leq i(2) < j(2) \leq \dots \leq i(2n) < j(2n) \leq m$ such that $d(U_{i(k)}, U_{j(k)}) \geq \rho$ for each $k = 1, 2, \dots, 2n$. By Lemma 4.3, we can conclude that one of the conditions (1) and (2) as in Lemma 4.4 is satisfied. \square

Lemma 4.5 [10, (3.2)]. *Let X be a set and let X_1, X_2, \dots, X_k be subsets of X such that $X = \bigcup_{j=1}^k X_j$. Then there exists a sufficiently large natural number $n(k) (> 2k^2)$ such that for any sequence $a_1, b_1, a_2, b_2, \dots, a_{n(k)}, b_{n(k)}$ of points of X , there exist $i_1 < i_2 < i_3$ such that a_{i_1} and a_{i_2} are contained in some X_s and b_{i_3} is contained in some X_t which contains b_{i_1} .*

Lemma 4.6. *Suppose that X is a continuum. Then there exists $\delta_1 > 0$ such that if $x, y \in X$, $x \neq y$, and \mathcal{U} is any finite open cover of X , then there exist an integer $N \geq 0$*

and $\eta > 0$ such that if $[V_1, V_2, \dots, V_m]$ is an η -chain from x to y , then $[f^N(V_1), f^N(V_2), \dots, f^N(V_m)]$ or $[f^{-N}(V_1), f^{-N}(V_2), \dots, f^{-N}(V_m)]$ is a refinement of \mathcal{U} , and is crooked between U_s and U_t , where $U_s, U_t \in \mathcal{U}$ and $d(U_s, U_t) \geq \delta_1 - 2 \cdot \text{mesh } \mathcal{U}$.

Proof. Put $k = \# \mathcal{U}$. Then Lemma 4.6 follows from Lemmas 4.4 and 4.5. \square

To prove Theorem 4.1, we also need the following. Let X be a tree-like continuum. For any subset M of $X \times X$, define $M^f = \{(x, y) \in X \times X \mid \text{for any } \gamma > 0 \text{ and any finite open cover } \mathcal{U} \text{ of } X \text{ such that the nerve } N(\mathcal{U}) \text{ is a tree, there exists } (x', y') \in M \text{ such that } x' \neq y' \text{ and there exists a finite open cover } \mathcal{V} \text{ of } X \text{ with mesh } \mathcal{V} < \gamma \text{ such that } \mathcal{V} \text{ is a refinement of } \mathcal{U}, \text{ the nerve } N(\mathcal{V}) \text{ is a tree and a chain } [V_1, V_2, \dots, V_m] \text{ from } x' \text{ to } y' \text{ of } \mathcal{V} \text{ is crooked between } U_x \text{ and } U_y, \text{ where } U_x \text{ and } U_y \text{ are elements of } \mathcal{U} \text{ such that } x \in U_x \text{ and } y \in U_y\}$ (see Fig. 2). By definition of M^f , we obtain the following propositions.

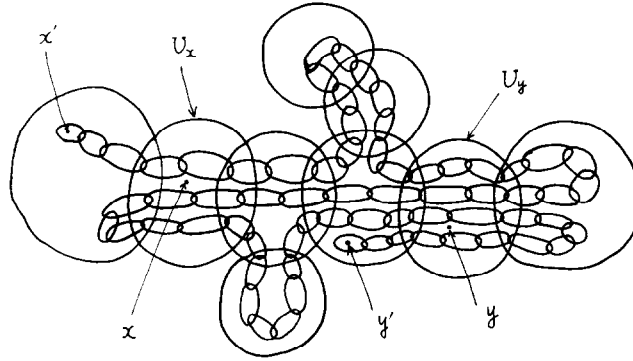


Fig. 2.

Proposition 4.7. M^f is closed in $X \times X$.

Proposition 4.8. $M^f \supset (M^f)^f$.

For a subset M of $X \times X$ and ordinal numbers, define $M_1 = M^f$, $M_{\alpha+1} = (M_\alpha)^f$, and $M_\lambda = \bigcap_{\alpha < \lambda} M_\alpha$, where λ is a limit ordinal number.

Theorem 4.9. If X is a hereditarily decomposable tree-like continuum, then $M_\alpha = \emptyset$ for some countable ordinal number α .

Proof. Note that $X \times X$ is separable. Since M_α is closed in $X \times X$ and $M_\alpha \supset M_\beta$ for $\alpha \leq \beta$, there is a countable ordinal α such that $M_\alpha = M_\beta$ for all $\beta \geq \alpha$. In particular, $(M_\alpha)^f = M_\alpha$. We shall show that $M_\alpha = \emptyset$. Suppose, on the contrary, that $M_\alpha \neq \emptyset$. Choose $(x_1, y_1) \in M_\alpha$. By the definition of M^f , we may assume that $x_1 \neq y_1$. Since X is a tree-like continuum, there is a finite open cover \mathcal{U}_1 of X such that

- (a.1) $\text{mesh } \mathcal{U}_1 < \min\{1/2, d(x_1, y_1)/3\}$, and
- (b.1) the nerve $N(\mathcal{U}_1)$ is a tree.

By induction, we can choose $(x_i, y_i) \in M_\alpha$ ($i = 1, 2, \dots$) and finite open covers \mathcal{U}_i of X such that

- (a.i) $\text{mesh } \mathcal{U}_i < 1/2^i$ and \mathcal{U}_{i+1} is a refinement of \mathcal{U}_i ,
- (b.i) the nerve $N(\mathcal{U}_i)$ is a tree, and
- (c.i) a chain $[U_1^{i+1}, U_2^{i+1}, \dots, U_{m(i+1)}^{i+1}]$ from x_{i+1} to y_{i+1} of \mathcal{U}_{i+1} is crooked between $U_{x_i}^i$ and $U_{y_i}^i$, where $x_i \in U_{x_i}^i \in \mathcal{U}_i$ and $y_i \in U_{y_i}^i \in \mathcal{U}_i$.

By (b.i) and (c.i), inductively, for each $i = 2, 3, \dots$, we can choose subchains $[U_{x_1}^i, \dots, U_{y_1}^i]$ of $[U_1^i, U_2^i, \dots, U_{m(i)}^i]$ such that

- (1) $U_{x_1}^1 \supset \text{Cl } U_{x_1}^2 \supset U_{x_1}^2 \supset \text{Cl } U_{x_1}^3 \supset \dots$,
- (2) $U_{y_1}^1 \supset \text{Cl } U_{y_1}^2 \supset U_{y_1}^2 \supset \text{Cl } U_{y_1}^3 \supset \dots$, and
- (3) $[U_{x_1}^{i+1}, \dots, U_{y_1}^{i+1}]$ is crooked between $U_{x_1}^i$ and $U_{y_1}^i$

(see Fig. 3). By (a.i), $\{\text{Cl } U_{x_1}^i\}_{i=1,2,\dots}$ (respectively $\{\text{Cl } U_{y_1}^i\}_{i=1,2,\dots}$) is convergent to a point x of $\text{Cl } U_{x_1}^1$ (respectively a point y of $\text{Cl } U_{y_1}^1$). Let H be the irreducible subcontinuum between x and y in X . Then we can conclude that H is nondegenerate and indecomposable. This is a contradiction. \square

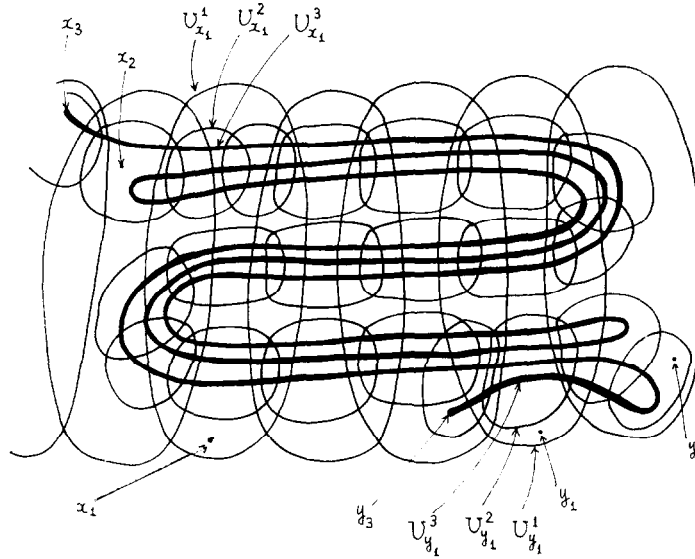


Fig. 3.

Proof of Theorem 4.1. Let X be a hereditarily decomposable tree-like continuum. Suppose, on the contrary, that there exists an expansive homeomorphism f on X . Put $M = X \times X$. According to Theorem 4.9, there exists a countable ordinal number α such that $M_\alpha = \emptyset$. Let $c > 0$ be an expansive constant for f and $0 < \varepsilon < c/2$. Choose $\delta_1 > 0$ satisfying the conditions as in Lemma 4.6. Let $x_0, y_0 \in X$ with $x_0 \neq y_0$. Since X is tree-like, there exists a sequence $\mathcal{U}_1, \mathcal{U}_2, \dots$ of finite open covers of X such that

- (1) the nerve $N(\mathcal{U}_i)$ is a tree,

- (2) \mathcal{U}_{i+1} is a refinement of \mathcal{U}_i , and
- (3) $\text{mesh } \mathcal{U}_i < 1/2^i$ for each $i = 1, 2, \dots$.

According to Lemma 4.6, we can choose a sequence $\mathcal{V}_1, \mathcal{V}_2, \dots$ of finite open covers of X such that for some integer $N(i) \in \mathbb{Z}$,

- (4) $f^{N(i)}(\mathcal{V}_i)$ is a refinement of \mathcal{U}_i , each $N(\mathcal{V}_i)$ is a tree, and
- (5) if $[V_1, V_2, \dots, V_m]$ is a chain from x_0 to y_0 of \mathcal{V}_i , then the chain $[f^{N(i)}(V_1), f^{N(i)}(V_2), \dots, f^{N(i)}(V_m)]$ is crooked between U_1^i and U_2^i , where U_1^i and U_2^i are elements of \mathcal{U}_i with $d(U_1^i, U_2^i) \geq \delta_1 - 2 \cdot \text{mesh } \mathcal{U}_i$.

Since X is compact, we may assume that $\{\text{Cl } U_1^i\}_{i=1,2,\dots}$ (respectively $\{\text{Cl } U_2^i\}_{i=1,2,\dots}$) is convergent to a point x_1 of X (respectively a point y_1 of X) (see (3)). Then by (5), $d(x_1, y_1) \geq \delta_1$. We obtain $(x_1, y_1) \in X \times X$.

For a countable ordinal number λ , we assume that we have obtained (x_α, y_α) of $X \times X$ for all $\alpha < \lambda$ as before. We will define $(x_\lambda, y_\lambda) \in X \times X$ recursively in the following way: Consider two cases.

Case 1: $\lambda = \alpha + 1$. By an argument similar to the above one, we can obtain $(x_{\alpha+1}, y_{\alpha+1}) \in X \times X$.

Case 2: λ is a limit ordinal number. In this case, take a sequence $\alpha_1 < \alpha_2 < \dots$ of countable ordinal numbers such that $\lim_{i \rightarrow \infty} \alpha_i = \lambda$. Then we may assume that $\{x_{\alpha_i}\}_{i=1,2,\dots}$ (respectively $\{y_{\alpha_i}\}_{i=1,2,\dots}$) is convergent to a point x_λ of X (respectively a point y_λ of X). Then $d(x_\lambda, y_\lambda) \geq \delta_1$. Hence we obtain $(x_\lambda, y_\lambda) \in X \times X$.

Next, we shall show that for each integer n and each countable ordinal number α , $(f^n(x_\alpha), f^n(y_\alpha)) \in M_\alpha$. We shall prove this fact by transfinite induction. First, we show that for each integer n , $(f^n(x_1), f^n(y_1)) \in M_1$. Let \mathcal{U} be any finite open cover of X such that the nerve $N(\mathcal{U})$ is a tree and let $\gamma > 0$. By the constructions of x_1 and y_1 , there exists \mathcal{U}_i such that $\text{Cl } U_1^i \subset U_{x_1}$, $\text{Cl } U_2^i \subset U_{y_1}$ and \mathcal{U}_i is a refinement of \mathcal{U} with $\text{mesh } \mathcal{U}_i < \gamma$, where U_{x_1} (respectively U_{y_1}) is an element of \mathcal{U} containing x_1 (respectively y_1). By (4) and (5), $f^{N(i)}(\mathcal{V}_i)$ is a refinement of \mathcal{U}_i and if $[f^{N(i)}(V_1), f^{N(i)}(V_2), \dots, f^{N(i)}(V_m)]$ is a chain from $f^{N(i)}(x_0)$ to $f^{N(i)}(y_0)$ of $f^{N(i)}(\mathcal{V}_i)$, then it is crooked between U_1^i and U_2^i . Hence $(f^{N(i)}(x_0), f^{N(i)}(y_0))$ satisfies the conditions of the definition of M^f , which implies that $(x_1, y_1) \in M^f = M_1$. Since f is a homeomorphism, we can conclude that $(f^n(x_1), f^n(y_1)) \in M_1$ for all integers $n \in \mathbb{Z}$.

Assume that λ is a countable ordinal number such that $(f^n(x_\alpha), f^n(y_\alpha))$ is contained in M_α for all $\alpha < \lambda$ and all integers $n \in \mathbb{Z}$. Consider two cases:

Case 1: $\lambda = \alpha + 1$. By the inductive assumption, $(f^n(x_\alpha), f^n(y_\alpha)) \in M_\alpha$. By an argument similar to the above one, we can conclude that $(f^n(x_{\alpha+1}), f^n(y_{\alpha+1})) \in M_{\alpha+1}$ for all integers $n \in \mathbb{Z}$.

Case 2: λ is a limit ordinal number. By an argument similar to the above one, we can prove that $(f^n(x_\lambda), f^n(y_\lambda)) \in M_\alpha$ for all $\alpha < \lambda$. This implies that

$$(f^n(x_\lambda), f^n(y_\lambda)) \in \bigcap_{\alpha < \lambda} M_\alpha = M_\lambda.$$

Consequently, we have proved that $M_\alpha \neq \emptyset$ for all countable ordinal numbers α . This is a contradiction. This completes the proof. \square

Corollary 4.10. *There are no expansive homeomorphisms on tree-like continua which have countable path-components.*

Proof. If a tree-like continuum X contains an indecomposable nondegenerate subcontinuum, then X has uncountable path-components. Hence Theorem 4.1 implies Corollary 4.10. \square

Corollary 4.11 [12]. *There are no expansive homeomorphisms on dendroids.*

Corollary 4.12. *There are no expansive homeomorphisms on hereditarily decomposable arc-like continua.*

5. The nonexistence of expansive homeomorphisms of hereditarily decomposable circle-like continua

A continuum X is said to be *circle-like* if for any $\varepsilon > 0$ there exists a finite open cover \mathcal{U} of X such that $\text{mesh } \mathcal{U} < \varepsilon$ and the nerve $N(\mathcal{U})$ is a simple closed curve. The p -adic solenoids ($p \geq 2$) are indecomposable circle-like continua which admit expansive homeomorphisms. In this section, we prove the following.

Theorem 5.1. *There are no expansive homeomorphisms on hereditarily decomposable circle-like continua.*

The proof is similar to one of Theorem 4.1, but we need slightly different considerations.

Outline of proof of Theorem 5.1. Let X be a hereditarily decomposable circle-like continuum. Suppose, on the contrary, that there exists an expansive homeomorphism f on X . For each point p of X , let g_p denote the intersection of all continua which contain interiorly a continuum that contains p interiorly (see the proof of [2, Theorem 8]). Then $\mathcal{G} = \{g_p\}$ is an upper semi-continuous collection of mutually exclusive continua filling the continuum X and \mathcal{G} is a simple closed curve with respect to its elements. In fact, by the proof of [8, (3.4)] there is a monotone map r from X onto a simple closed curve S (i.e., $r^{-1}(y)$ is connected for each $y \in S$). Combining this fact and Bing's theorem [2, Theorem 8], we can easily see that \mathcal{G} is a simple closed curve. Let $\pi: X \rightarrow \mathcal{G} = S$ be the quotient map. By the definition of π , we can see that there exists a homeomorphism $h: S \rightarrow S$ such that the following

diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \pi \downarrow & & \downarrow \pi \\ S & \xrightarrow{h} & S \end{array}$$

Since S admits no expansive homeomorphisms [7], for some $s_0 \in S$, $\pi^{-1}(s_0)$ is nondegenerate. Put

$$F = \{(x, y) \in X \times X \mid \pi(x) = \pi(y)\}.$$

For any subset M of F , define $M^f = \{(x, y) \in F \mid \text{for any } \gamma > 0 \text{ and for any finite open cover } \mathcal{U} \text{ of } X \text{ such that the nerve } N(\mathcal{U}) \text{ is a simple closed curve, there exists } (x', y') \in M \text{ such that } x' \neq y' \text{ and there exists a finite open cover } \mathcal{V} \text{ of } X \text{ with mesh } \mathcal{V} < \gamma \text{ such that } \mathcal{V} \text{ is a refinement of } \mathcal{U}, \text{ the nerve } N(\mathcal{V}) \text{ is a simple closed curve, } \{V \in \mathcal{V} \mid V \cap \pi^{-1}(\pi(x')) \neq \emptyset\} \text{ is a refinement of } \{U \in \mathcal{U} \mid U \cup \pi^{-1}(\pi(x)) \neq \emptyset\}, \text{ and a chain } [V_1, V_2, \dots, V_m] \text{ from } x' \text{ to } y' \text{ of } \{V \in \mathcal{V} \mid V \cup \pi^{-1}(\pi(x')) \neq \emptyset\} \text{ is crooked between } U_x \text{ and } U_y, \text{ where } U_x \text{ and } U_y \text{ are elements of } \mathcal{U} \text{ such that } x \in U_x \text{ and } y \in U_y\}.$

We may assume that the nerves $N(\{V \in \mathcal{V} \mid V \cap \pi^{-1}(\pi(x')) \neq \emptyset\})$ and $N(\{U \in \mathcal{U} \mid U \cap \pi^{-1}(\pi(x)) \neq \emptyset\})$ are one-point sets or arcs, because π is a monotone map. By the definition, we see that M^f is closed in F and $(M^f)^f \subset M^f$. As before, define $M_1 = M^f$, $M_{\alpha+1} = (M_\alpha)^f$ and $M_\lambda = \bigcap_{\alpha < \lambda} M_\alpha$, where λ is a limit ordinal number. By an argument similar to the proof of Theorem 4.1, we can prove that if $M_\alpha \neq \emptyset$ for any countable ordinal number α , then for some $s \in S$, $\pi^{-1}(s)$ contains an indecomposable nondegenerate subcontinuum. Hence $M_\alpha = \emptyset$ for some countable ordinal number α .

Let $M = F$. Let $c > 0$ be an expansive constant for f and $0 < \varepsilon < c/2$. Choose δ_1 satisfying the conditions as in Lemma 4.6. Let $x_0, y_0 \in \pi^{-1}(s_0)$ with $x_0 \neq y_0$. Since X is circle-like, there is a sequence $\mathcal{U}_1, \mathcal{U}_2, \dots$ of finite open covers of X such that

- (1) $N(\mathcal{U}_i)$ is a simple closed curve,
- (2) \mathcal{U}_{i+1} is a refinement of \mathcal{U}_i , and
- (3) $\text{mesh } \mathcal{U}_i < 1/2^i$ for each $i = 1, 2, \dots$

According to Lemma 4.6, we can choose a sequence $\mathcal{V}_1, \mathcal{V}_2, \dots$ of finite open covers of X such that for some integers $N(i) \in \mathbb{Z}$,

- (4) $N(\mathcal{V}_i)$ is a simple closed curve and $f^{N(i)}(\mathcal{V}_i)$ is a refinement of \mathcal{U}_i ,
- (5) for $\mathcal{V}_i(s_0) = \{V \in \mathcal{V}_i \mid V \cap \pi^{-1}(s_0) \neq \emptyset\}$ and $\mathcal{U}_i(h^{N(i)}(s_0)) = \{U \in \mathcal{U}_i \mid U \cap \pi^{-1}(h^{N(i)}(s_0)) \neq \emptyset\}$, $f^{N(i)}(\mathcal{V}_i(s_0))$ is a refinement of $\mathcal{U}_i(h^{N(i)}(s_0))$, and
- (6) if $[V_1, V_2, \dots, V_m]$ is a chain from x_0 to y_0 of $\mathcal{V}_i(s_0)$, then $[f^{N(i)}(V_1), f^{N(i)}(V_2), \dots, f^{N(i)}(V_m)]$ is crooked between U_1^i and U_2^i , where U_1^i and U_2^i are elements of $\mathcal{U}_i(h^{N(i)}(s_0))$ and $d(U_1^i, U_2^i) \geq \delta_1 - 2 \cdot \text{mesh } \mathcal{U}_i$.

Since X is compact, we may assume that $\lim_{i \rightarrow \infty} \text{Cl } U_1^i = x_1$ and $\lim_{i \rightarrow \infty} \text{cl } U_2^i = y_1$. Then by (5) and (6), we see that $(x_1, y_1) \in F$ and $d(x_1, y_1) \geq \delta_1$. By an argument similar to the above one, we obtain $(x_\alpha, y_\alpha) \in F$ for all countable ordinal numbers

α such that $d(x_\alpha, y_\alpha) \geq \delta_1$. Also, by an argument similar to the above one, we can conclude that $(f^n(x_\alpha), f^n(y_\alpha)) \in M_\alpha$ for all countable ordinal numbers α and all integers $n \in \mathbb{Z}$. This is a contradiction. This completes the proof. \square

Corollary 5.2. *There are no expansive homeomorphisms on circle-like continua which have countable path-components.*

The following problems remain open.

Problem 1. Does there exist a 1-dimensional Peano continuum admitting an expansive homeomorphism? In particular, is there an expansive homeomorphism on the Menger's Universal Curve?

Problem 2. If a continuum X admits an expansive homeomorphism, does X contain an indecomposable (nondegenerate) subcontinuum?

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